

# $p$ -adic meromorphic functions $f'P'(f)$ , $g'P'(g)$ sharing a small function

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## Abstract

Let  $\mathbb{K}$  be a complete algebraically closed  $p$ -adic field of characteristic zero. Let  $f, g$  be two transcendental meromorphic functions in the whole field  $\mathbb{K}$  or meromorphic functions in an open disk that are not quotients of bounded analytic functions. Let  $P$  be a polynomial of uniqueness for meromorphic functions in  $\mathbb{K}$  or in an open disk and let  $\alpha$  be a small meromorphic function with regards to  $f$  and  $g$ . If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  counting multiplicity, then we show that  $f = g$  provided that the multiplicity order of zeroes of  $P'$  satisfy certain inequalities. If  $\alpha$  is a Moebius function or a non-zero constant, we can obtain more general results on  $P$ .

## Introduction and Main Results

Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero, complete for an ultrametric absolute value denoted by  $|\cdot|$ . We denote by  $\mathcal{A}(\mathbb{K})$  the  $\mathbb{K}$ -algebra of entire functions in  $\mathbb{K}$ , by  $\mathcal{M}(\mathbb{K})$  the field of meromorphic functions in  $\mathbb{K}$ , i.e. the field of fractions of  $\mathcal{A}(\mathbb{K})$  and by  $\mathbb{K}(x)$  the field of rational functions.

Let  $a \in \mathbb{K}$  and  $R \in ]0, +\infty[$ . We denote by  $d(a, R)$  the closed disk  $\{x \in \mathbb{K} : |x - a| \leq R\}$  and by  $d(a, R^-)$  the open disk  $\{x \in \mathbb{K} : |x - a| < R\}$ . We denote by  $\mathcal{A}(d(a, R^-))$  the set of analytic functions in  $d(a, R^-)$ , i.e. the  $\mathbb{K}$ -algebra of power series  $\sum_{n=0}^{\infty} a_n(x - a)^n$  converging in  $d(a, R^-)$  and by  $\mathcal{M}(d(a, R^-))$  the field of meromorphic functions inside  $d(a, R^-)$ , i.e. the field of fractions of  $\mathcal{A}(d(a, R^-))$ . Moreover, we denote by  $\mathcal{A}_b(d(a, R^-))$  the  $\mathbb{K}$ -subalgebra of  $\mathcal{A}(d(a, R^-))$  consisting of the bounded analytic functions in  $d(a, R^-)$ , i.e. which satisfy  $\sup_{n \in \mathbb{N}} |a_n|R^n < +\infty$ . And we denote by  $\mathcal{M}_b(d(a, R^-))$  the field of fractions of  $\mathcal{A}_b(d(a, R^-))$ . Finally, we denote by  $\mathcal{A}_u(d(a, R^-))$  the set of unbounded analytic functions in  $d(a, R^-)$ , i.e.  $\mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$ . Similarly, we set  $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$ .

The problem of value sharing a small function by functions of the form  $f'P'(f)$  was examined first when  $P$  was just of the form  $x^n$  [7], [18], [24]. More recently it was examined when  $P$  was a polynomial such that  $P'$  had exactly two distinct zeroes [15], [17], [20], both in complex analysis and in  $p$ -adic analysis. In [15], [17] the functions were meromorphic on  $\mathbb{C}$ , with a small function that was a constant or the identity. In [20], the problem was considered for analytic functions

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in the field  $\mathbb{K}$ : on one hand for entire functions and on the other hand for unbounded analytic functions in an open disk.

Here we consider functions  $f, g \in \mathcal{M}(\mathbb{K})$  or  $f, g \in \mathcal{M}(d(a, R^-))$  and ordinary polynomials  $P$ : we must only assume certain hypotheses on the multiplicity order of the zeroes of  $P'$ . The method for the various theorems we will show is the following: assuming that  $f'P'(f)$  and  $g'P'(g)$  share a small function, we first prove that  $f'P'(f) = g'P'(g)$ . Next, we derive  $P(f) = P(g)$ . And then, when  $P$  is a polynomial of uniqueness for the functions we consider, we can conclude  $f = g$ .

Now, in order to define small functions, we have to briefly recall the definitions of the classical Nevanlinna theory in the field  $\mathbb{K}$  and a few specific properties of ultrametric analytic or meromorphic functions.

Let  $\log$  be a real logarithm function of base  $> 1$  and let  $f \in \mathcal{M}(\mathbb{K})$  (resp.  $f \in \mathcal{M}(d(0, R^-))$ ) having no zero and no pole at 0. Let  $r \in ]0, +\infty[$  (resp.  $r \in ]0, R[$ ) and let  $\gamma \in d(0, r)$ . If  $f$  has a zero of order  $n$  at  $\gamma$ , we put  $\omega_\gamma(f) = n$ . If  $f$  has a pole of order  $n$  at  $\gamma$ , we put  $\omega_\gamma(f) = -n$  and finally, if  $f(\gamma) \neq 0, \infty$ , we put  $\omega_\gamma(f) = 0$ .

We denote by  $Z(r, f)$  the *counting function of zeroes of  $f$  in  $d(0, r)$* , counting multiplicity, i.e. we set

$$Z(r, f) = \sum_{\omega_\gamma(f) > 0, |\gamma| \leq r} \omega_\gamma(f) (\log r - \log |\gamma|).$$

In the same way, we set  $N(r, f) = Z\left(r, \frac{1}{f}\right)$  to denote the *counting function of poles of  $f$  in  $d(0, r)$* , counting multiplicity.

For  $f \in \mathcal{M}(d(0, R^-))$  having no zero and no pole at 0, the *Nevanlinna function* is defined by  $T(r, f) = \max \{Z(r, f) + \log |f(0)|, N(r, f)\}$ .

Now, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

**Definition.** Let  $f \in \mathcal{M}(\mathbb{K})$  (resp. let  $f \in \mathcal{M}(d(0, R^-))$ ) such that  $f(0) \neq 0, \infty$ . A function  $\alpha \in \mathcal{M}(\mathbb{K})$  (resp.  $\alpha \in \mathcal{M}(d(0, R^-))$ ) having no zero and no pole at 0 is called a *small function with respect to  $f$* , if it satisfies  $\lim_{r \rightarrow +\infty} \frac{T(r, \alpha)}{T(r, f)} = 0$  (resp.  $\lim_{r \rightarrow R^-} \frac{T(r, \alpha)}{T(r, f)} = 0$ ).

If 0 is a zero or a pole of  $f$  or  $\alpha$ , we can make a change of variable such that the new origin is not a zero or a pole for both  $f$  and  $\alpha$ . Thus it is easily seen that the last relation does not really depend on the origin.

We denote by  $\mathcal{M}_f(\mathbb{K})$  (resp.  $\mathcal{M}_f(d(0, R^-))$ ) the set of small meromorphic functions with respect to  $f$  in  $\mathbb{K}$  (resp. in  $d(0, R^-)$ ).

Let us remember the following definition.

**Definition.** Let  $f, g, \alpha \in \mathcal{M}(\mathbb{K})$  (resp. let  $f, g, \alpha \in \mathcal{M}(d(0, R^-))$ ). We say that  $f$  and  $g$  *share the function  $\alpha$  C.M.*, if  $f - \alpha$  and  $g - \alpha$  have the same zeroes with the same multiplicity in  $\mathbb{K}$  (resp. in  $d(0, R^-)$ ).

Recall that a polynomial  $P \in \mathbb{K}[x]$  is called a *polynomial of uniqueness* for a class of functions  $\mathcal{F}$  if for any two functions  $f, g \in \mathcal{F}$  the property  $P(f) = P(g)$  implies  $f = g$ .

Actually, in a p-adic field, we can obtain various results, not only for functions defined in the whole field  $\mathbb{K}$  but also for functions defined inside an open disk because the p-adic Nevanlinna Theory works inside a disk, for functions of  $\mathcal{M}_u(d(0, R^-))$ .

We can now state our main theorems on the problem  $f'P'(f)$ ,  $g'P'(g)$  share a small function.

**Theorem 1.** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$ , let  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{K}^*$ ,  $l \geq 2$ ,  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l-1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$\begin{aligned} n &\geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2), \\ n &\geq k + 2, \\ \text{if } l = 2, &\text{ then } n \neq 2k, 2k + 1, 3k + 1, \\ \text{if } l = 3, &\text{ then } n \neq 2k + 1, 3k_i - k \ \forall i = 2, 3. \end{aligned}$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem 2.** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$ , let  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{K}^*$ ,  $l \geq 2$ ,  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l-1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$\begin{aligned} n &\geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2), \\ n &\geq k + 2, \\ \text{if } l = 2, &\text{ then } n \neq 2k, 2k + 1, 3k + 1, \\ \text{if } l = 3, &\text{ then } n \neq 2k + 1, 3k_i - k \ \forall i = 2, 3. \end{aligned}$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha$  be a Moebius function. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem 3.** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$ , let  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{K}^*$ ,  $l \geq 2$ ,  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l-1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$\begin{aligned} n &\geq k + 2, \\ n &\geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2). \end{aligned}$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha$  be a non-zero constant. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem 4.** Let  $a \in K$  and  $R > 0$ . Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}_u(d(a, R^-))$  and let  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{K}^*$ ,  $l \geq 2$ ,  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l-1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$\begin{aligned} n &\geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2), \\ n &\geq k + 3, \\ \text{if } l = 2, &\text{ then } n \neq 2k, 2k + 1, 3k + 1, \\ \text{if } l = 3, &\text{ then } n \neq 2k + 1, 3k_i - k \ \forall i = 2, 3. \end{aligned}$$

Let  $f, g \in \mathcal{M}_u(d(a, R^-))$  and let  $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem 5.** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$  such that  $P'$  is of the form

$$b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^*, \text{ satisfying:}$$

$$n \geq l + 10,$$

$$\text{if } l = 3, \text{ then } n \neq 2l - 1.$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem 6.** Let  $a \in K$  and  $R > 0$ . Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}_u(d(a, R^-))$  such

that  $P'$  is of the form  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)$  with  $l \geq 3, b \in \mathbb{K}^*$  satisfying:

$$n \geq l + 10,$$

$$\text{if } l = 3, \text{ then } n \neq 2l - 1.$$

Let  $f, g \in \mathcal{M}_u(d(a, R^-))$  and let  $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem 7.** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$  such that  $P'$  is of the form

$$P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^* \text{ satisfying}$$

$$n \geq l + 9,$$

$$\text{if } l = 3, \text{ then } n \neq 2l - 1.$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha$  be a Moebius function. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem 8.** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$  such that  $P'$  is of the form

$$P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^* \text{ satisfying } n \geq l + 9.$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha$  be a non-zero constant. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem 9.** Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  be non-identically zero. Let  $a \in \mathbb{K} \setminus \{0\}$ . If  $f'f^n(f - a)$  and  $g'g^n(g - a)$  share the function  $\alpha$  C.M. and if  $n \geq 12$ ,

then either  $f = g$  or there exists  $h \in \mathcal{M}(\mathbb{K})$  such that  $f = \frac{a(n+2)}{n+1} \left( \frac{h^{n+1} - 1}{h^{n+2} - 1} \right) h$  and  $g = \frac{a(n+2)}{n+1} \left( \frac{h^{n+1} - 1}{h^{n+2} - 1} \right)$ . Moreover, if  $\alpha$  is a constant or a Moebius function, then the conclusion holds whenever  $n \geq 11$ .

Inside an open disk, we have a version similar to the general case in the whole field.

**Theorem 10.** Let  $f, g \in \mathcal{M}_u(d(0, R^-))$ , and let  $\alpha \in \mathcal{M}_f(d(0, R^-)) \cap \mathcal{M}_g(d(0, R^-))$  be non-identically zero. Let  $a \in \mathbb{K} \setminus \{0\}$ . If  $f'f^n(f - a)$  and  $g'g^n(g - a)$  share the function  $\alpha$  C.M. and

$n \geq 12$ , then either  $f = g$  or there exists  $h \in \mathcal{M}(d(0, R^-))$  such that  $f = \frac{a(n+2)}{n+1} \left( \frac{h^{n+1} - 1}{h^{n+2} - 1} \right) h$  and  $g = \frac{a(n+2)}{n+1} \left( \frac{h^{n+1} - 1}{h^{n+2} - 1} \right)$ .

## Specific Lemmas

**Lemma 1.** Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental (resp.  $f, g \in \mathcal{M}_u(d(0, R^-))$ ). Let  $P(x) = x^{n+1}Q(x)$  be a polynomial such that  $n \geq \deg(Q) + 2$  (resp.  $n \geq \deg(Q) + 3$ ). If  $P'(f)f' = P'(g)g'$  then  $P(f) = P(g)$ .

**Lemma 2.** Let  $Q(x) = (x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i} \in \mathbb{K}[x]$  ( $a_i \neq a_j, \forall i \neq j$ ) with  $l \geq 2$  and  $n \geq \max\{k_2, \dots, k_l\}$  and let  $k = \sum_{i=2}^l k_i$ . Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental (resp.  $f, g \in \mathcal{M}_u(d(0, R^-))$ ) such that  $\theta = Q(f)f'Q(g)g'$  is a small function with respect to  $f$  and  $g$ . We have the following :

If  $l = 2$  then  $n$  belongs to  $\{k, k+1, 2k, 2k+1, 3k+1\}$ .

If  $l = 3$  then  $n$  belongs to  $\{\frac{k}{2}, k+1, 2k+1, 3k_2-k, \dots, 3k_l-k\}$ .

If  $l \geq 4$  then  $n = k+1$ .

If  $\theta$  is a constant and  $f, g \in \mathcal{M}(\mathbb{K})$  then  $n = k+1$ .

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